Proof of Simpson's Rule

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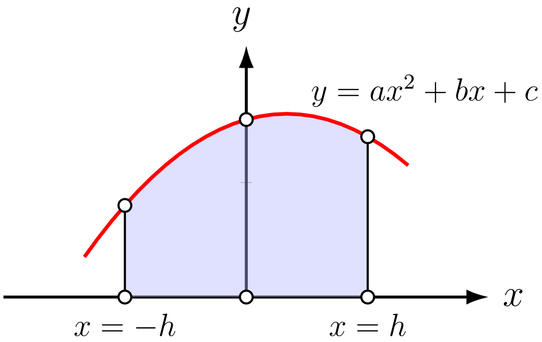
(July 24, 2021)

We consider the area under the curve $y = f(x)$ where f is a general parabola:

$$\begin{aligned} f : \mathbb{R} &\longrightarrow \mathbb{R} \\ x &\longmapsto f(x) = ax^2 + bx + c \end{aligned}$$

for some $a, b, c \in \mathbb{R}$ with $a \neq 0$. For simplicity, we consider the area under this curve on the interval $[-h, h]$ for some $h > 0$:



The curve passes through the points $(-h, y_0)$, $(0, y_1)$ and (h, y_2) where

$$\begin{aligned} y_0 &= ah^2 - bh + c \\ y_1 &= c \\ y_2 &= ah^2 + bh + c. \end{aligned}$$

Considering these as 3 equations in 3 unknowns a, b, c, we can solve to find

$$\begin{aligned} a &= \frac{1}{2h^2}(y_0 - 2y_1 + y_2) \\ b &= \frac{1}{2h}(y_2 - y_0) \\ c &= y_1. \end{aligned} \tag{1}$$

The area under the curve A (shaded in the diagram above) is given by the definite integral:

$$\begin{aligned} A &= \int_{x=-h}^{h} (ax^2 + bx + c)\, dx = \frac{1}{3}ax^3 + \frac{1}{2}bx^2 + cx \Big|_{x=-h}^{h} \\ &= \frac{2}{3}ah^3 + 2ch = \frac{h}{3}(2ah^2 + 6c). \end{aligned}$$

or after substituting the values of a, c from (1):

$$A = \frac{h}{3}(y_0 + 4y_1 + y_2).$$

Now consider the integral of a function F:

$$\int_{x=a}^{b} F(x)\,dx, \tag{2}$$

where F is assumed to be continuous on $[a, b]$. We divide this interval into n **even** subintervals of equal length:

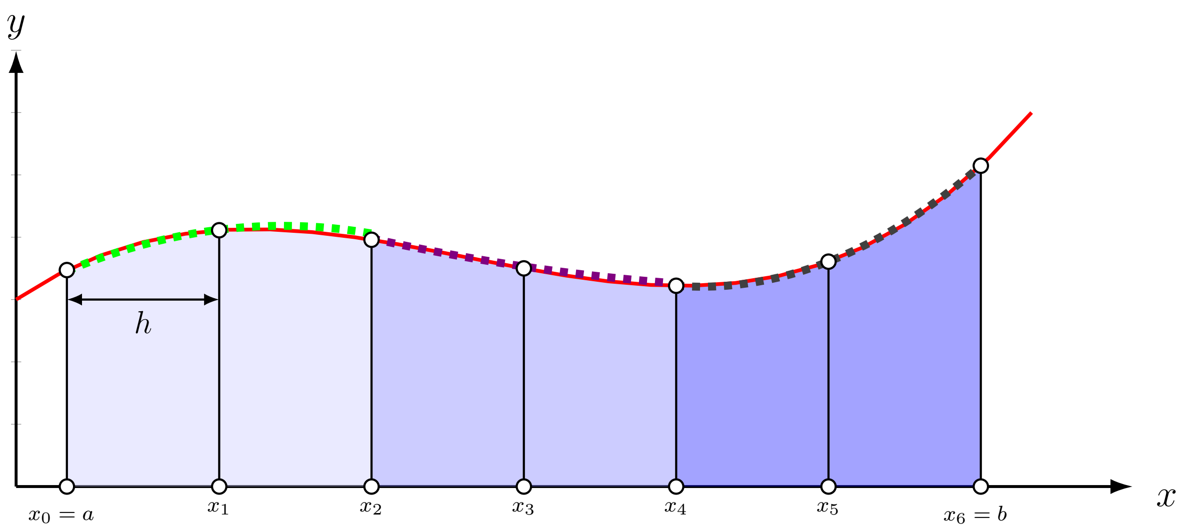
$$h = \frac{b-a}{n},$$

and introduce the $n+1$ points (or ordinates):

$$x_0 = a,\ x_1 = a+h,\ x_2 = a+2h,\ \ldots,\ x_{n-1} = x+(n-1)h,\ x_n = a+nh = b.$$

We evaluate the function F at these points:

$$y_0 = F(x_0),\ y_1 = F(x_1),\ y_2 = F(x_2),\ \ldots,\ y_{n-1} = F(x_{n-1}),\ y_n = F(x_n).$$



Example of applying Simpson's rule to a function using 6 intervals. The dashed curves denote the quadratic interpolating polynomials between each three successive points.

Since a quadratic polynomial can be used to interpolate three distinct points, we can approximate (2) by adding the areas under the parabolic arcs through three successive points:

$$\begin{aligned}\int_{x=a}^{b} F(x)\,dx &\approx \frac{h}{3}(y_0+4y_1+y_2)+\frac{h}{3}(y_2+4y_3+y_4)+\cdots+\frac{h}{3}(y_{n-2}+4y_{n-1}+y_n)\\ &= \frac{h}{3}\left(y_0+4y_1+2y_2+4y_3+\cdots+4y_{n-1}+y_n\right).\end{aligned}$$

This is known as *Simpson's rule*.